The Ising and Spin O(n) models - proofs

Thursday, 24 June 2021 13:07

The spin O(n) model ($n \geq 1$ integer)

Let $\mathbb{T}_L^d$ be the graph with vertex set $\Lambda = \{-L+1, -L, \ldots, L\}^d$ and edges between vertices differing in exactly one coordinate, where they differ by 1 modulo $2L$.

Configurations: $\Omega := \{\sigma : \Lambda \to S^{n-1}\}$.

probability measure: $\mu_{\mathbb{T}_L^d, n, \beta}$ with density

β = inverse temperature

$\langle \cdot, \cdot \rangle$ = standard inner product in $\mathbb{R}^n$.

$$\frac{1}{Z} e^{\beta \sum_{u \sim v} \langle \sigma_u, \sigma_v \rangle}$$

with respect to $d\sigma$, the product uniform measure on each coordinate. Here, $Z = \int_\Omega e^{\beta \sum_{u \sim v} \langle \sigma_u, \sigma_v \rangle} d\sigma$

We study the correlation

$$\rho_{xy} := \mathbb{E} \langle \sigma_x, \sigma_y \rangle$$

for distant $x, y \in \Lambda$.

Last time: We saw $\rho_{xy} \leq C e^{-c \|x-y\|_1}$

graph distance in $\mathbb{T}_L^d$

when either $d=1$ or $d \geq 2$ and β is sufficiently small.

Reference: Lectures on the spin and loop O(n) models / Peled - Spinka.

Low-temperature Ising model - the Peierls argument

Low-temper[illegible] argument

Goal: For $h=1$ (Ising model, configurations take values in $\{-1,1\}$), for $d \geq 2$, when β is sufficiently large,

$$\rho_{x,y} \geq c_{d,\beta} > 0$$

uniformly in $x, y \in \Lambda$ and system size L.

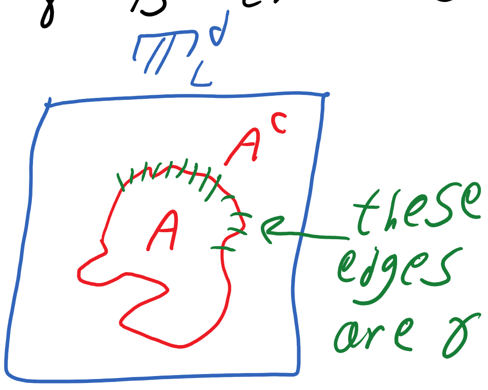
Proof: First rewrite

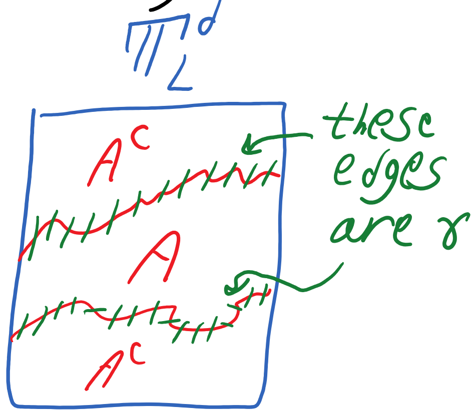
$$\rho_{x,y} = \mathbb{E}(\langle \sigma_x, \sigma_y \rangle) \underset{\sigma_x, \sigma_y \in \{-1,1\}}{=} \mathbb{E}(\sigma_x \sigma_y) = \mathbb{P}(\sigma_x = \sigma_y) - \mathbb{P}(\sigma_x \neq \sigma_y)$$

$$= 1 - 2\mathbb{P}(\sigma_x \neq \sigma_y).$$

So, we need to that $\mathbb{P}(\sigma_x \neq \sigma_y)$ is small when β is large.

Contour: A contour γ is a set of edges in $\mathbb{T}_L^d$ such that there exists a set of vertices $A \subseteq V(\mathbb{T}_L^d)$ with both A and A^c connected in $\mathbb{T}_L^d$ $(A, A^c \neq \emptyset)$ s.t. γ is the edge boundary of A.





A^c is still connected since $\mathbb{T}_L^d$ is a torus

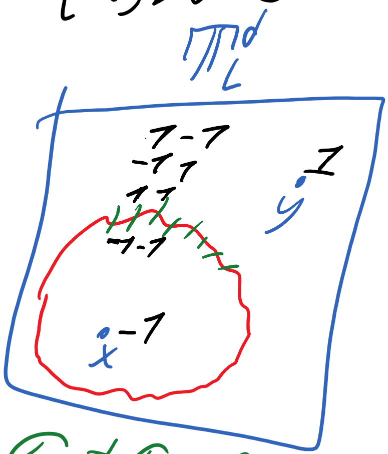
We say that γ separates x, y if any path since $\mathbb{T}^d_L$ is a torus

We say that γ separates x, y if any path in $\mathbb{T}^d_L$ from x to y must cross γ.

We write $|\gamma|$ for the number of edges in γ.

interface (or domain wall): A contour γ is an interface for $\sigma \in \Omega$ if γ separates x and y and

$$\sigma_u \neq \sigma_v \quad \text{for all } \{u,v\} \in \gamma.$$



$\sigma_u \neq \sigma_v$ for every edge in γ.

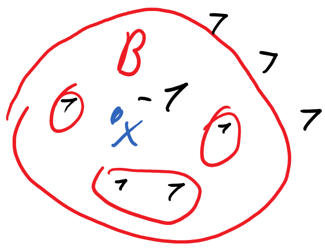
Deterministic fact:

If $\sigma \in \Omega$ satisfies

$$\sigma_x \neq \sigma_y$$

then there exists some interface for σ.

Indeed, let B be the connected component of x in $\{z \in V(\mathbb{T}^d_L) : \sigma_z = \sigma_x\}$.



Now let A be the connected comp. of y in B^c. Then A and A^c are connected and the edge bdry. of A is an interface.

Thus, to reach our goal, it suffices to bound $\mathbb{P}(\exists \text{ an interface for } \sigma)$ when β is large.

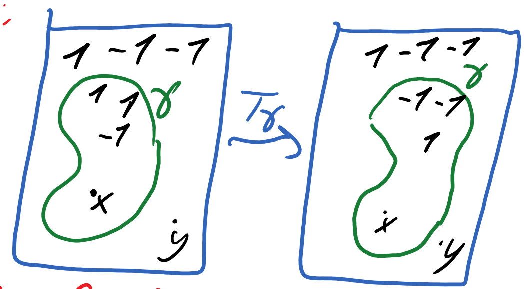
Fixed interface: Let γ be a contour separating x and y.

Fixed interface:

Let γ be a contour separating x and y.

$$P(\gamma \text{ is an interface}) = \frac{\sum\limits_{\substack{\sigma\in\Omega \\ \gamma \text{ is an interface}}} e^{\beta\sum_{u\sim v}\sigma_u\sigma_v}}{\sum\limits_{\sigma\in\Omega} e^{\beta\sum_{u\sim v}\sigma_u\sigma_v}} \leq$$

Peierls bound is based on a transformation T_γ:



T_γ flips the spins on the side of x of γ.

Note, T_γ acts 1-1 on Ω.

$$\leq \frac{\sum\limits_{\substack{\sigma\in\Omega \\ \gamma \text{ is an interface for } \sigma}} e^{\beta\sum_{u\sim v}\sigma_u\sigma_v}}{\sum\limits_{\substack{\sigma\in\Omega \\ \gamma \text{ is an interface for } \sigma}} e^{\beta\sum_{u\sim v}T_\gamma(\sigma)_u T_\gamma(\sigma)_v}}$$

$$= e^{-2\beta|\gamma|}$$

Note, if $\sigma\in\Omega$ has γ as an interface then

$$\sum_{u\sim v}\sigma_u\sigma_v - \sum_{u\sim v}T_\gamma(\sigma)_u T_\gamma(\sigma)_v = -2|\gamma|.$$

Consequently,

$$P(\exists \text{ an interface for } \sigma) \overset{\text{union bound}}{\leq} \sum_{k=1}^{\infty}\sum_{\substack{\gamma \text{ contour} \\ \text{separating} \\ x \text{ and } y \\ |\gamma|=k}} P(\gamma \text{ is an interface})$$

$$\overset{\text{previous bound}}{\leq} \sum_{k=1}^{\infty} e^{-2\beta k}\,\left|\{\gamma : \gamma \text{ contour separating } x \text{ and } y,\ |\gamma|=k\}\right|.$$

Counting contours:

Claim: $\exists\, C$ depending only on the dimension d s.t.

$|\{\gamma \text{ contour separating} \ldots\}| \leq e^{C(d)\cdot k}$

dimension ...

$$\left|\left\{\gamma : \begin{matrix}\gamma \text{ contour separating} \\ x \text{ and } y,\ |\gamma| = k\end{matrix}\right\}\right| \le e^{C(d)\cdot k}.$$

We will not prove this

Thus, when $\beta >> C(d)$ we get from the previous bound that $\mathbb{P}\left(\begin{matrix}\exists \text{ an interface} \\ \text{for } \sigma\end{matrix}\right)$ is small.

No continuous-symmetry breaking in two dimensions - the Mermin-Wagner theorem

Fix now $d=2$ and let $n \ge 2$

spins in the unit circle, the unit sphere ...

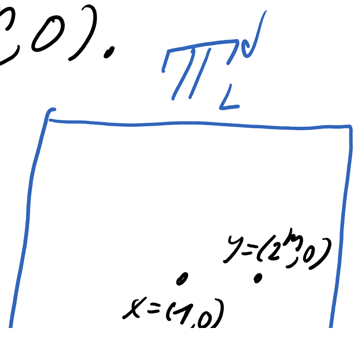
Thm.: At any $\beta \in [0,\infty)$,

$$\rho_{x,y} \le \frac{C_{n,\beta}}{\|x-y\|_1^{c_{n,\beta}}}.$$

In particular, unlike for the Ising model there is never long-range order $\rho_{x,y} \ge c_{n,\beta}$.

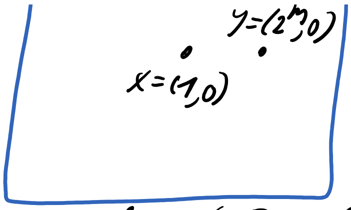
The power-law decay rate is due to McBryan-Spencer (1977).

For notational simplicity, we focus on the case $n=2$ (spins on S^1) and $x=(1,0)$ and $y=(2^m,0)$.



Let us first explain why we cannot have $\rho_{x,y} \ge c_{n,\beta}$.

This is the essence of



This is the essence of the result.

We express the configuration σ in terms of angles. We may write each $\sigma_v \in S^1$ as a point in the complex plane, as $\sigma_v = e^{i\theta_v}$ with $\theta_v \in [-\pi, \pi)$.

Then the prob. density of θ is

$$\frac{1}{Z} e^{\beta \sum_{u\sim v} \langle \sigma_u, \sigma_v \rangle} = \frac{1}{Z} e^{\beta \sum_{u \sim v} \cos(\theta_u - \theta_v)}$$

wrt. product uniform measure.

Let θ be sampled from this measure.

Let $\alpha : V(\mathbb{T}_L^d) \to \mathbb{R}$ be a fixed function.

Define $\theta^+ := \theta + \alpha \mod 2\pi$.

Claim: $\underbrace{d_{TV}}_{\text{Total variation distance}}(\theta, \theta^+) \le \frac{1}{2}\sqrt{\beta}\left(\sum_{u\sim v} (\alpha_u - \alpha_v)^2\right)^{1/2}$.

Proof: By Pinsker's inequality,

$$d_{TV}(\theta, \theta^+) \le \sqrt{\frac{1}{2} d_{KL}(\theta \| \theta^+)}.$$

Here $d_{KL}(\theta \| \theta^+) = \mathbb{E}_\theta \left(\operatorname{Log}\left(\frac{d\mathcal{L}(\theta)}{\underbrace{d\mathcal{L}(\theta^+)}_{\text{relative density of } \theta \text{ with respect to } \theta^+}} \right)\right) =$

$$= \mathbb{E}_\theta \left(\operatorname{Log}\left(\frac{e^{\beta \sum_{u\sim v} \cos(\theta_u - \theta_v)}}{e^{\beta \sum_{u \sim v} \cos(\theta_u - \alpha_u - (\theta_v - \alpha_v))}} \right)\right) =$$

$$= \mathbb{E}_\theta\left(\beta \sum_{u\sim v} \cos(\theta_u-\theta_v) - \cos(\theta_u-\alpha_u-(\theta_v-\alpha_v))\right) =$$

Taylor expansion: $\cos(\varphi+\gamma) = \cos(\varphi) - \sin(\varphi)\cdot\gamma - \frac{1}{2}\cos(\varphi')\gamma^2$
where φ' is between φ and $\varphi+\gamma$.

$$= \mathbb{E}_\theta\left(\beta \sum_{u\sim v} -\sin(\theta_u-\theta_v)(\alpha_u-\alpha_v) + \frac{1}{2}\cos(\text{something})\cdot(\alpha_u-\alpha_v)^2\right)$$

$\theta_u-\theta_v \overset{d}{=} \theta_v-\theta_u$

$$= \beta\,\mathbb{E}_\theta\left(\sum_{u\sim v} \frac{1}{2}\cos(\text{something})(\alpha_u-\alpha_v)^2\right) \le$$

$$\le \frac{1}{2}\beta \sum_{u\sim v}(\alpha_u-\alpha_v)^2.$$

How to use the above Claim?

Note that $\rho_{xy} = \mathbb{E}(\langle\sigma_x,\sigma_y\rangle) = \mathbb{E}\cos(\theta_x-\theta_y)$.

To prove that ρ_{xy} is small it is sufficient to show that $\theta_x-\theta_y$ modulo 2π doesn't concentrate much prob. in any short interval.

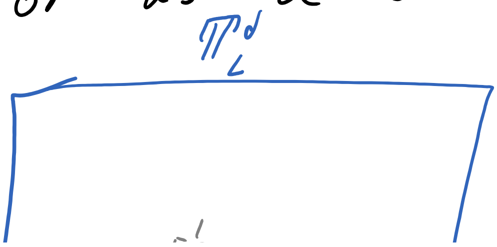
To this end we construct $\alpha: V(\mathbb{T}_L^d)\to\mathbb{R}$ with $\alpha_x=0$, α_y large and

$$\sqrt{\beta}\sum_{u\sim v}(\alpha_u-\alpha_v)^2 \text{ small}.$$

This is only possible when $d=2$.

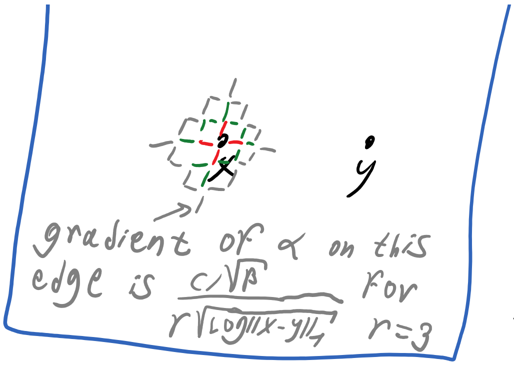
The optimal α is harmonic off x and y, but for us a suboptimal choice suffices.

E.g.



We take

$$\alpha_u-\alpha_v = \frac{C/\sqrt{\beta}}{r\sqrt{\log\|x-y\|_1}}$$



$\alpha_u - \alpha_v = r\sqrt{\log\|x-y\|_1}$ when the edge $\{u,v\}$ is at distance r from x with $1 \le r \le \|x-y\|_1$ and with u on the side of y.

Then $\sqrt{\beta}\sum_{u\sim v}(\alpha_u-\alpha_v)^2 \le C$

and $\alpha_y \approx \frac{C}{\sqrt{\beta}}\sqrt{\log\|x-y\|_1}$.

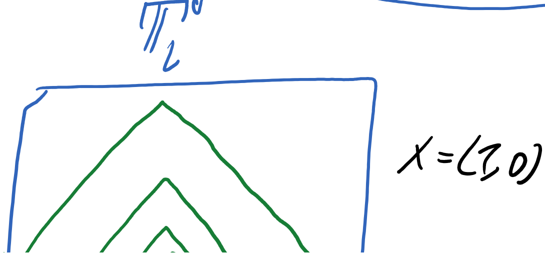
With such a fcn., θ and $\theta^+ = \theta+\alpha \mod 2\pi$ have a similar dist. by the claim, but $\theta_x - \theta_y$ differs significantly from $\theta^+_x - \theta^+_y$. This implies that $\theta_x - \theta_y$ does not concentrate much prob. on any short interval,

so $\rho_{x,y} = \mathbb{E}\cos(\theta_x - \theta_y)$ is small.

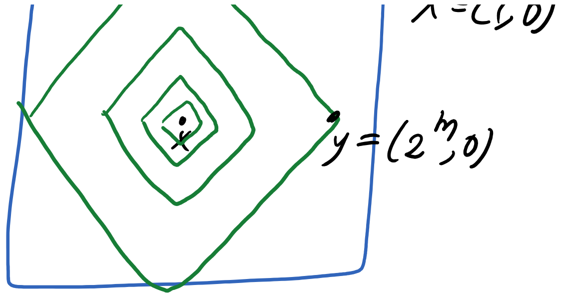
With this approach we can show that $\rho_{x,y}$ tends to zero as $\|x-y\|_1 \to \infty$ (uniformly in L) but we don't get the power-law decay.

Additional trick for power-law decay (hybrid of Dobrushin-Shlosman and Pfister's approaches

Divide $\mathbb{T}^d_L$ into layers according to the distance $\|u\|_1$ of a vertex



according to the ...
$\|u\|_1$ of a vertex
from the origin.



For each layer consider the gradients of the angles θ: $\nabla\theta_{=\ell} := \left(\theta_u - \theta_v : \|u\|_1, \|v\|_1 = 2^\ell\right)$.

Given the information in $\nabla\theta_{=\ell}$, the only degree of freedom for the angles in the layer is a global rotation.

Observe that

$$\rho_{x,y} = \mathbb{E}\cos(\theta_y - \theta_x) = \mathbb{E}\, e^{i(\theta_y - \theta_x)} =$$

$$= \mathbb{E}\left(\prod_{\ell=0}^{m-1} e^{i(\theta_{(2^{\ell+1},0)} - \theta_{(2^\ell,0)})}\right) =$$

$$= \mathbb{E}\left[\mathbb{E}\left(\prod_{\ell=0}^{m-1} e^{i(\theta_{(2^{\ell+1},0)} - \theta_{(2^\ell,0)})} \,\middle|\, \begin{array}{l}\nabla\theta_{=\ell} \\ \text{for } 0\le\ell\le m-1\end{array}\right)\right]$$

Fact:
Cond. on $(\nabla\theta_{=\ell} : 0\le\ell\le m-1)$, the angle differences $\left(\theta_{(2^{\ell+1},0)} - \theta_{(2^\ell,0)}\right)_{\ell=0}^{m-1}$ become independent!

Exercise: Convince yourself of this!

$$= \mathbb{E}\left[\prod_{\ell=0}^{m-1} \mathbb{E}\left[e^{i(\theta_{(2^{\ell+1},0)} - \theta_{(2^\ell,0)})} \,\middle|\, \begin{array}{l}\nabla\theta_{=\ell} \\ \text{for } 0\le\ell\le m-1\end{array}\right]\right]$$

It remains only to show that each of the factors in the product is at most 1 ... to get the

of the factors in the product is at most $1-\varepsilon_\beta$, to get the power-law decay of $\rho_{x,y}$.
The bound $1-\varepsilon_\beta$ can be obtained with the technique we used to show $\rho_{x,y}$ tends to zero as $\|x-y\|_1 \to \infty$.
We proved the power-law decay for $n=2$.
For $n>2$, can simply write each σ_v as $(\sigma_v^1, \sigma_v^2, \ldots, \sigma_v^n)$ and run the above argument after conditioning on $\sigma_v^3, \ldots, \sigma_v^n$ for all v.

Long-range order in dimensions $d \geq 3$ - the infra-red bound (Fröhlich-Simon-Spencer 76)

In dimensions $d \geq 3$, for any $n \geq 1$,
Thm.: There exists β_1 s.t. if $\beta > \beta_1$ then
$$\frac{1}{|V(\mathbb{T}_L^d)|^2} \sum_{x,y \in V(\mathbb{T}_L^d)} \rho_{x,y} \geq c_{d,n,\beta}.$$

Ideas from the Proof: $\Lambda = V(\mathbb{T}_L^d)$

Gaussian domination: Define, for $\tau: \Lambda \to \mathbb{R}^n$,
$$W(\tau) := \exp\Big(-\frac{\beta}{2} \sum_{u \sim v} \|\tau_u - \tau_v\|_2^2\Big)$$
(Recall that the density of the spin $O(n)$ model is $\frac{1}{\ }e^{\beta \sum_{u\sim v} \langle \sigma_u, \sigma_v \rangle}$.

Recall that the ... model is $\frac{1}{Z} e^{\beta \sum_{u \sim v} \langle \sigma_u, \sigma_v \rangle}$.

There is a close connection since

$$W(\sigma) = C(|\Lambda|)\, e^{\beta \sum_{u\sim v} \langle \sigma_u, \sigma_v \rangle}$$

$\|\sigma_u\|_2 = 1$ for every u

Define $$Z(\tau) := \int_{\Omega} W(\sigma + \tau)\, d\sigma.$$

Thm. (Gaussian domination): For any $\tau : \Lambda \to \mathbb{R}^n$,

$$Z(\tau) \leq Z(0).$$

Equivalently, the bound may be stated as

over the spin $O(n)$ model $\to$ $$\mathbb{E}\left(\frac{W(\sigma+\tau)}{W(\sigma)}\right) = \frac{Z(\tau)}{Z(0)} \leq 1, \ \forall \tau : \Lambda \to \mathbb{R}^n$$

To establish this is the difficult part of the proof. It is proved via Reflection positivity and the proof relies heavily on the specific choice of $\mathbb{T}_L^d$ (the torus geometry).

It is unclear how to obtain similar inequalities for other geometries. Also unclear how to prove for a more general class of models.

... bound: we introduce a

class of models.

The infra-red bound: We introduce a convenient notation. Define the discrete Laplacian operator Δ just a matrix acting on $\mathbb{C}^\Lambda$ by

$$(\Delta f)_u := \sum_{v : v \sim u} (f_v - f_u)$$

Denote the inner product on $\mathbb{C}^\Lambda$ by $(f,g) := \sum_{v\in\Lambda} f_v \overline{g_v}$.

Discrete Green identity: For all $f,g \in \mathbb{C}^\Lambda$,

$$\sum_{u\sim v} (f_u - f_v)\overline{(g_u - g_v)} = (f, -\Delta g).$$

(in short, $(\nabla f, \nabla g) = (f, -\Delta g)$)

Extend the notation to act on functions $f, g \in (\mathbb{C}^n)^\Lambda$ by acting coordinate by coordinate.

Then $W(\tau) = e^{-\frac{\beta}{2}\sum_{u\sim v} \|\tau_u - \tau_v\|_2^2}$

$$= e^{-\frac{\beta}{2}(\nabla\tau, \nabla\tau)} = e^{-\frac{\beta}{2}(\tau, -\Delta\tau)}$$

So the above ineq. $\mathbb{E}\left(\frac{W(\sigma+\tau)}{W(\sigma)}\right) \le 1$

becomes

$$\mathbb{E}\left(e^{-\frac{\beta}{2}(\sigma+\tau, (-\Delta)(\sigma+\tau)) + \frac{\beta}{2}(\sigma, -\Delta\sigma)}\right) \le 1$$

$$\Leftarrow\Rightarrow \mathbb{E}\left(e^{\beta(\sigma, \Delta\tau)}\right) \le e^{\frac{1}{2}\beta(\tau, -\Delta\tau)}, \quad \forall \tau: \Lambda \to \mathbb{R}^n.$$

This bounds an exponential moment. Replacing τ by $\varepsilon \cdot \tau$ for a small ε and taking the Taylor expansion of the exponential, we get a bound on a second moment: $\mathbb{E}((\sigma, \Delta\tau)^2) \le \frac{1}{\beta}(\tau, -\Delta\tau)$.

The idea now is to choose specific test functions τ. It is convenient to take the eigenvectors of Δ, which are the Fourier vectors.

The rest of the proof is as follows: We express σ in the Fourier basis. Parseval's equality says that the sum of squares of the Fourier coefficients is large. But the above ineq. bounds all Fourier coef. except the one corresponding to the average of σ. We deduce in $d \ge 3$ and large β that the square of the average of σ is large $\Rightarrow$ Long-range order.

is large ⟹ Long-range order.